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SUBJECT: Some Properties of Quaternions Related to Euler's Theorem
Case 620

DATE: June 26, 1970

FROM: B. D. Elrod

ABSTRACT

A quaternion is a generalized complex number with four elements. This memorandum summarizes several properties of quaternions scattered through the literature which are related to the kinematical problem in spacecraft attitude control. The results provide background for understanding the "strapdown" calculations to be used for workshop attitude determination during the Skylab mission.

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MEMORANDUM FOR FILE

1.0 INTRODUCTION

The quaternion invented by Hamilton (1843) provides a convenient formulation for Euler's Theorem which is the basis for the kinematical relationships involved in rotational motion. In general, a quaternion (q) is defined as the hypercomplex number

$$q = q_0 + i\, q_1 + j\, q_2 + k\, q_3 = q_0 + \underline{q} \tag{1}$$

with the properties

$$\begin{aligned} ij &= -ji = k & \qquad -ik &= ki = j \\ jk &= -kj = i & \qquad i^2 &= j^2 = k^2 = -1 \end{aligned} \tag{2}$$

The complex conjugate of q is

$$q^* \equiv q_0 - \underline{q} \tag{3}$$

Also q is a unit quaternion if

$$qq^* = q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1 \tag{4}$$

Hence the inverse (q^{-1}) of a unit quaternion is its complex conjugate, $q^{-1} = q^*$.

Alternatively, q can be regarded as a 4 element vector where

$$q = \begin{Bmatrix} \underline{q} \\ q_0 \end{Bmatrix}, \qquad \underline{q} = \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} \tag{5}$$

with special rules, analogous to (2) applicable for quaternion products. The corresponding inverse is

$$q^{-1} = \begin{Bmatrix} -\underline{q} \\ q_0 \end{Bmatrix} \tag{6}$$

2.0 RELATION TO EULER'S THEOREM

Euler's Theorem states that the general displacement of a rigid body with one point fixed is a rotation (θ) about some axis ($\underline{e}$). (See Figure 1.) If e_1, e_2, e_3 define components

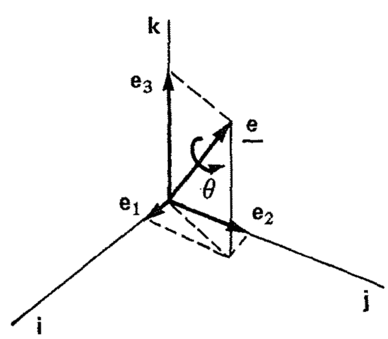

FIGURE 1 - EIGENAXIS AND EULER ROTATION ANGLE

of the unit vector $\underline{e}$ (eigenaxis) along (i j k), the four parameters devised by Euler (1776) for defining body orientation are

$$\begin{aligned} \xi &= e_1 \sin(\theta/2) \\ \eta &= e_2 \sin(\theta/2) \\ \zeta &= e_3 \sin(\theta/2) \\ \chi &= \cos(\theta/2) \end{aligned} \tag{7}$$

the quaternion formulation is then

$$q = \chi + i\xi + j\eta + k\zeta = \cos(\theta/2) + (ie_1 + je_2 + ke_3)\sin(\theta/2) \tag{8}$$

or in vector form

$$q = \begin{Bmatrix} \underline{q} \\ q_0 \end{Bmatrix} = \begin{Bmatrix} \underline{e} \sin(\theta/2) \\ \cos(\theta/2) \end{Bmatrix} \tag{9}$$

3.0 QUATERNION PRODUCTS

As given by Euler, the result of two successive rotations (θ_1, then θ_2) about respective eigenaxes ($\underline{e}_1$, $\underline{e}_2$) is represented by the quaternion product

$$q = q_2\, q_1 \tag{10}$$

where the elements of q_2 and q_1 are <u>both</u> referred to the original coordinate frame. In general the product of two quaternions

$$r = r_0 + \underline{r} \tag{11}$$

and

$$p = p_0 + \underline{p} \tag{12}$$

can be written as

$$m = m_0 + \underline{m} = pr = (p_0\, r_0 - \underline{p}\cdot\underline{r}) + (\underline{p}\times\underline{r} + p_0\, \underline{r} + r_0\, \underline{p}) \tag{13}$$

or for reverse order

$$n = n_0 + \underline{n} = rp = (p_0\, r_0 - \underline{p}\cdot\underline{r}) + (-\underline{p}\times\underline{r} + p_0\, \underline{r} + r_0\, \underline{p}) \tag{14}$$

The effect of inverting order of the product is a change in sign of the term $\underline{p}\times\underline{r}$.

The quaternion product can be expressed in matrix form as

$$m = pr = \begin{Bmatrix} \underline{m} \\ m_0 \end{Bmatrix} = M_p \begin{Bmatrix} \underline{r} \\ r_0 \end{Bmatrix} = \hat{M}_r \begin{Bmatrix} \underline{p} \\ p_0 \end{Bmatrix} \tag{15}$$

and

$$\mathcal{n} = \mathcal{r}\mathcal{p} = \begin{Bmatrix} \underline{n} \\ n_0 \end{Bmatrix} = \hat{M}_{\mathcal{p}} \begin{Bmatrix} \underline{r} \\ r_0 \end{Bmatrix} = M_{\mathcal{r}} \begin{Bmatrix} \underline{p} \\ p_0 \end{Bmatrix} \tag{16}$$

where*

$$M_{\mathcal{p}} = \left[\begin{array}{c|c} p_0 E + \tilde{\underline{p}} & \underline{p} \\ \hline -\underline{p}^T & p_0 \end{array}\right], \qquad \hat{M}_{\mathcal{p}} = \left[\begin{array}{c|c} p_0 E - \tilde{\underline{p}} & \underline{p} \\ \hline -\underline{p}^T & p_0 \end{array}\right] \tag{17}$$

and $M_{\mathcal{r}}$ and $\hat{M}_{\mathcal{r}}$ are defined analogously.

Consider the triple product $\mathcal{p}^{-1}\mathcal{n} = \mathcal{p}^{-1}\mathcal{r}\mathcal{p}$. From (13) through (17) it follows that

$$\mathcal{p}^{-1}\mathcal{n} = M_{\mathcal{p}^{-1}} \begin{pmatrix} \underline{n} \\ n_0 \end{pmatrix} = \left[\begin{array}{c|c} p_0 E - \tilde{\underline{p}} & -\underline{p} \\ \hline \underline{p}^T & p_0 \end{array}\right] \begin{Bmatrix} \underline{n} \\ n_0 \end{Bmatrix} = \begin{Bmatrix} -\tilde{\underline{p}}\,\underline{n} + p_0 \underline{n} - n_0 \underline{p} \\ p_0\, n_0 + \underline{p}^T \underline{n} \end{Bmatrix} \tag{18}$$

and

$$\mathcal{r}\mathcal{p} = \mathcal{n} = \begin{pmatrix} \underline{n} \\ n_0 \end{pmatrix} = \begin{Bmatrix} -\tilde{\underline{p}}\,\underline{r} + p_0 \underline{r} + r_0 \underline{p} \\ p_0\, r_0 - \underline{p}^T \underline{r} \end{Bmatrix} \tag{19}$$

---

*E is a 3×3 unit matrix, T represents the transpose operation and ∿ over a vector reflects the cross product operation (i.e., $\tilde{\underline{p}}$ is a 3×3 matrix and $\tilde{\underline{p}}\,\underline{p} = \underline{0}$).

Thus*

$$p^{-1} r p = \begin{Bmatrix} -\tilde{\underline{p}}(-\tilde{\underline{p}}\underline{r} + p_0\underline{r} + r_0\underline{p}) + p_0(-\tilde{\underline{p}}\underline{r} + p_0\underline{r} + r_0\underline{p}) - (p_0 r_0 - \underline{p}^T\underline{r})\underline{p} \\ p_0(p_0 r_0 - \underline{p}^T\underline{r}) + \underline{p}^T(-\tilde{\underline{p}}\underline{r} + p_0\underline{r} + r_0\underline{p}) \end{Bmatrix}$$

$$= \begin{Bmatrix} [\tilde{\underline{p}}\,\tilde{\underline{p}} + p_0^2 E - 2p_0\,\tilde{\underline{p}} + \underline{p}\,\underline{p}^T]\underline{r} - r_0 \cancelto{0}{\tilde{\underline{p}}\,\underline{p}} \\ r_0(p_0^2 + \underline{p}^T\underline{p}) - \cancelto{0}{\underline{p}^T\tilde{\underline{p}}}\,\underline{r} \end{Bmatrix} \tag{20}$$

4.0 COORDINATE TRANSFORMATIONS

Hamilton has shown that a vector rotation can be expressed as a quaternion product

$$v' = q\ v\ q^{-1} \tag{21}$$

where** $v'=\underline{v}'$, $v=\underline{v}$ and q is the quaternion associated with the vector rotation from $\underline{v}$ to $\underline{v}'$. Both $\underline{v}'$ and $\underline{v}$ are expressed in the same coordinate frame.

The analogous form representing coordinate transformation is

$$x^2 = q^{-1}\ x^1\ q \tag{22}$$

where $x^1=\underline{x}^1$ and $x^2=\underline{x}^2$ define coordinate axes in coordinate frames 1 and 2. A superscript 1 may be added to q to emphasize that it is expressed relative to frame 1.

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*E is a 3×3 unit matrix, T represents the transpose operation and ∿ over a vector reflects the cross product operation (i.e., $\tilde{\underline{p}}$ is a 3×3 matrix and $\tilde{\underline{p}}\ \underline{p} = \underline{0}$).

**Three dimensional vectors can be regarded as quaternions with zero real part (i.e., $v_0'=v_0=0$).

The relationship with the direction cosine matrix (A) can be seen after relating (22) to (20) with $p \equiv q$ expressed as in (9) and $\mathscr{r} \equiv x^1 = \underline{x}^1$ $(\therefore r_0 = x_0 = 0)$. Then,

$$x^2 = \begin{Bmatrix} \underline{x}^2 \\ 0 \end{Bmatrix} = q^{-1}\, x^1\, q = \begin{Bmatrix} [\tilde{\underline{q}}\,\tilde{\underline{q}} + q_0^2 E - 2q_0\,\tilde{\underline{q}} + \underline{q}\,\underline{q}^T]\underline{x}^1 \\ 0 \end{Bmatrix} = \begin{Bmatrix} A\underline{x}^1 \\ 0 \end{Bmatrix}$$

or

$$\underline{x}^2 = A\,\underline{x}^1 \tag{23}$$

Substituting elements of $\underline{q}$ from (1) into A yields

$$A = \tilde{\underline{q}}\,\tilde{\underline{q}} + q_0^2\, E - 2q_0\,\tilde{\underline{q}} + \underline{q}\,\underline{q}^T \tag{24}$$

$$= \begin{bmatrix} (q_1^2 - q_2^2 - q_3^2 + q_0^2) & 2(q_1\, q_2 + q_3\, q_0) & 2(q_1\, q_3 - q_2\, q_0) \\ 2(q_1\, q_2 - q_3\, q_0) & (q_2^2 - q_1^2 - q_3^2 + q_0^2) & 2(q_2\, q_3 + q_1\, q_0) \\ 2(q_1\, q_3 + q_2\, q_0) & 2(q_2\, q_3 - q_1\, q_0) & (q_3^2 - q_1^2 - q_2^2 + q_0^2) \end{bmatrix}$$

Expanding A with $\underline{q}$ and q_0 as in (9) yields*

$$A = \tilde{\underline{e}}\,\tilde{\underline{e}}\, s^2(\theta/2) + c^2(\theta/2)E - 2\, s(\theta/2)\, c(\theta/2)\tilde{\underline{e}} + \underline{e}\,\underline{e}^T\, s^2(\theta/2)$$

$$= [c^2(\theta/2) - s^2(\theta/2)]E + [\tilde{\underline{e}}\,\tilde{\underline{e}} + E + \underline{e}\,\underline{e}^T]s^2(\theta/2) - \tilde{\underline{e}}\, s\theta \tag{25}$$

$$= c\theta\, E + (1 - c\theta)\,\underline{e}\,\underline{e}^T - \tilde{\underline{e}}\, s\theta$$

which is the generic form of A in terms of $\underline{e}$ and θ.[3] Thus, A can be calculated from either (24) or (25) depending on whether $(\underline{q}, q_0)$ or $(\underline{e}, \theta)$ is given.

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*Note the identity: $E + \tilde{\underline{e}}\,\tilde{\underline{e}} = \underline{e}\,\underline{e}^T$. For brevity, $\sin(\;) \equiv s(\;)$ and $\cos(\;) \equiv c(\;)$.

In dynamical problems $(\underline{q}, q_0)$ can be determined from the quaternion differential equation [2,5,6]

$$\dot{q} = \frac{1}{2} q \ \Omega \tag{26}$$

where

$$\Omega = \begin{Bmatrix} \underline{\omega} \\ 0 \end{Bmatrix} \tag{27}$$

represents the angular velocity of coordinate frame 2 relative to frame 1. The matrix form of (26) is

$$\begin{Bmatrix} \dot{\underline{q}} \\ \dot{q}_0 \end{Bmatrix} = \frac{1}{2} \hat{M}_\Omega \begin{Bmatrix} \underline{q} \\ q_0 \end{Bmatrix} = \frac{1}{2} \left[\begin{array}{c|c} -\tilde{\underline{\omega}} & \underline{\omega} \\ \hline -\underline{\omega}^T & 0 \end{array} \right] \begin{Bmatrix} \underline{q} \\ q_0 \end{Bmatrix} \tag{28}$$

This result offers the advantage of solving only 4 differential equations instead of 6 as normally required in solving for elements of A from $\dot{A} = -\tilde{\underline{\omega}} A$ and orthogonality constraints on A.

5.0 ALTERNATE QUATERNION PRODUCT FORMULATION

In (10) the quaternion product

$$q = q_2 \ q_1 \tag{29}$$

represents two successive rotations (θ_1, θ_2) about eigenaxes $(\underline{e}_1, \underline{e}_2)$. In terms of coordinate transformations this represents transformation from frame 1 to frame 2, then from frame 2 to frame 3. Also q_2 and q_1 must be expressed in frame 1. To account for the rotational sequence and the frame to which quaternions are referred, (29) is rewritten as

$$q_{13}^1 = q_{23}^1 \ q_{12}^1 \tag{30}$$

where the superscript indicates quaternion frame of reference and the subscripts, the rotational order. In general

$$q^1_{1n} = q^1_{n-1,n} \; .... \; q^1_{23} \; q^1_{12} \tag{31}$$

The problem with this formulation for handling successive rotations is that all quaternions must be expressed in frame 1, which is not always convenient. This can be circumvented as follows. Consider

$$x^3 = (q^1_{13})^{-1} \; x^1(q^1_{13}) = (q^1_{12})^{-1} \; (q^1_{23})^{-1} \; x^1(q^1_{23})(q^1_{12}) \tag{32}$$

and

$$x^3 = (q^2_{23})^{-1} \; x^2(q^2_{23}) \tag{33}$$

Since

$$x^2 = (q^1_{12})^{-1} \; x^1(q^1_{12}) \tag{34}$$

it follows that (33) is

$$x^3 = (q^2_{23})^{-1} \; (q^1_{12})^{-1} \; x^1(q^1_{12})(q^2_{23}) \tag{35}$$

Comparing (32) and (35) yields

$$(q^1_{12})(q^2_{23}) = (q^1_{23})(q^1_{12})$$

or

$$q^1_{23} = (q^1_{12})(q^2_{23})(q^1_{12})^{-1} \tag{36}$$

Thus, (30) can be written as

$$q^1_{13} = (q^1_{12})(q^2_{23})(q^1_{12})^{-1} \; (q^1_{12}) = (q^1_{12})(q^2_{23}) \tag{37}$$

and in general

$$q_{1n}^{1} = (q_{12}^{1}) \cdots (q_{n-1,n}^{n-1}) \tag{38}$$

This permits each quaternion in a product to be expressed relative to the local frame in which the rotation is made.

6.0 INTERCHANGEABILITY OF QUATERNIONS IN QUATERNION PRODUCTS

From the quaternion product formulations in (15) - (17) it is evident that a choice can be made on the order of terms in the matrix-vector product corresponding to $q_2\, q_1$. This property is useful when a sequence of quaternion products contains elements which vary at widely different rates, e.g.,

$$q_{14}^{1} = q_{34}^{1}\, q_{23}^{1}\, q_{12}^{1} \tag{39}$$

where q_{34}^{1} is constant, q_{12}^{1} varies slowly and q_{23}^{1} varies more rapidly. As discussed by Ickes [4] a substantial computational time saving can be realized by organizing the quaternion product such that

$$q_{14}^{1} = M\, q_{23}^{1} \tag{40}$$

where M is a quaternion coefficient matrix associated with q_{12}^{1} and q_{34}^{1}, which need not be updated continuously, only at intervals. M is obtained by writing (39) as in (15) and (17) with the same notational convention for the coefficient matrices and quaternions. Thus

$$q_{14}^{1} = M_{34}^{1}\, q_{13}^{1} = M_{34}^{1}\, \hat{M}_{12}^{1}\, q_{23}^{1} = M\, q_{23}^{1} \tag{41}$$

or also

$$q_{14}^{1} = \hat{M}_{12}^{1}\, q_{24}^{1} = \hat{M}_{12}^{1}\, M_{34}^{1}\, q_{23}^{1} = M\, q_{23}^{1} \tag{42}$$

so that

$$M = M_{34}^{1} \hat{M}_{12}^{1} = \hat{M}_{12}^{1} M_{34}^{1} \tag{43}$$

which is Ickes result with all q_{ij} and M_{ij} referred to frame 1.

For the alternate formulation of a quaternion product, as in (38), with all q_{ij} expressed in a local frame, a corresponding result can be obtained.

$$q_{14}^{1} = q_{12}^{1} q_{23}^{2} q_{34}^{3} = M_{12}^{1} \hat{M}_{34}^{3} q_{23}^{2} \tag{44}$$

or

$$q_{14}^{1} = \hat{M}_{34}^{3} M_{12} q_{23}^{2} \tag{45}$$

so that

$$M = M_{12}^{1} \hat{M}_{34}^{3} = \hat{M}_{34}^{3} M_{12} \tag{46}$$

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